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INTERVAL-VALUED INTUITIONISTIC HESITANT FUZZY EINSTEIN GEOMETRIC AGGREGATION OPERATORS

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ABSTRACT

Aggregation of fuzzy information in hesitant fuzzy environment is a new branch of hesitant fuzzy set (HFS) theory. HFS theory introduced by Torra and Narukowa has attracted significant interest from researchers in recent years. In this paper, we investigate the interval valued intuitionistic hesitant fuzzy (IVIHF) aggregation operators with the help of Einstein operations. First some new operations such as Einstein sum, Einstein product, and Einstein scalar multiplication on the interval valued intuitionistic hesitant fuzzy elements **(**IVIHFEs) are introduced. Then, some IVIHF aggregation operators such as interval valued intuitionistic hesitant fuzzy Einstein weighted geometric ($IVIHFWG^{\epsilon}$) operators and the interval valued intuitionistic hesitant fuzzy Einstein ordered weighted geometric ($IVIHFOWG^{\epsilon}$) operator are developed. Some of the properties of IVIHFEs are discussed in detail.

KEYWORDS: Einstein Operations, Hesitant Fuzzy Set, Interval Valued Intuitionistic Hesitant Fuzzy Elements, Interval Valued Intuitionistic Hesitant Fuzzy Einstein Weighted Geometric ($IVIHFWG^{\epsilon}$) Operators

I. INTRODUCTION

Fuzzy Set Theory by Zadeh [1] has been extended to several theories such as Atanassov's intuitionistic fuzzy set (AIFS) theory [2]. AIFSs is further generalized by Atanassov and Gargov [3] to accommodate the membership and non-membership functions to assume interval values, thereby introducing the concept of interval-valued intuitionistic fuzzy sets (IVIFSs). This extension mixes imprecision and hesitation. Recently, Torra and Narukawa [4] and Torra [5] proposed the hesitant fuzzy set (HFS), which is another generalization form of fuzzy set. The characteristic of HFS is that it allows membership degree to have a set of possible values. Therefore, HFS is a very useful tool in the situations where there are some difficulties in determining the membership of an element to a set. Lately, research on aggregation methods and multiple attribute decision making theories under hesitant fuzzy environment is very active. Xia et al [6] developed hesitant fuzzy aggregation operators. Combining the heronian mean and hesitant fuzzy sets, some new hesitant fuzzy Heronian mean (HFHM) operators are explored in [7].

Aggregation operators are essential mathematical tool for fuzzy decision-making. This tool is extended to the interval valued intuitionistic hesitant fuzzy environment. All aggregation operators introduced previously are based on the algebraic product and algebraic sum of intuitionistic fuzzy values (IFVs) or hesitant fuzzy elements (HFEs) to carry out the combination process. The algebraic operations algebraic product and algebraic sum are not the unique operations that can be used to perform the intersection and union. Einstein product and Einstein sum are good alternatives for they typically give the same smooth approximation as algebraic product and algebraic sum. For intuitionistic fuzzy information, Wang and Liu [8] developed some new intuitionistic fuzzy aggregation operators with the help of Einstein operations. There is little investigation on aggregation techniques using the Einstein operations to aggregate interval valued intuitionistic hesitant fuzzy information. Therefore, it is necessary to develop some interval valued intuitionistic hesitant fuzzy information aggregation operators based on Einstein operations.

In this paper we provide a novel extension to the IVIHFS setting which preserves the main properties of the usual aggregation operator. The focus of this paper is to investigate some properties of IVIHFEs based on Einstein operational laws and develop interval valued intuitionistic hesitant fuzzy Einstein aggregation operators. This paper is structured as follows. In Section 1, we give an introduction of the research background. In Section 2, we briefly review some basic concepts related to the IVIHFEs. In Section 3, we introduce some Einstein operations of IVIHFEs and analyze some desirable properties of the proposed operations. In Section 4, we develop some novel aggregating operators, such as the interval valued intuitionistic hesitant fuzzy Einstein weighted geometric (IVIHFWG$^{\varepsilon}$) operator, the interval valued intuitionistic fuzzy Einstein ordered weighted geometric (IVIHFOWG$^{\varepsilon}$) operator.

2. PRELIMINARIES

The concept of FS was extended to IFS [2] which is characterized by a membership function and a non-membership function.

Definition 2.1 IFS [2]

Let X be a fixed set. An IFS A in X is defined as $A=\{(x,\mu_A(x),\gamma_A(x)/x\in X)\}$ where μ_A and γ_A are mappings from X to the closed interval [0, 1] such that $0\le\mu_A(x)\le 1,\ 0\le\gamma_A(x)\le 1$ and for all $x\in X$,

$0\le\mu_A(x)+\gamma_A(x)\le 1$ and they denote respectively the degree of membership and degree of non-membership of element $x\in X$ to the set A.

Sometimes, instead of exact values a range of values may be a more appropriate measurement to represent the vagueness. Atanassov and Gargov [3] introduced the Interval Valued Intuitionistic Fuzzy Sets (IVIFS)

Definition 2.2 [3]

Let D [0, 1] be the set of all closed sub-intervals of [0, 1], an Interval Valued Intuitionistic Fuzzy Set $\tilde{A}$ in X is defined as $\tilde{A}=\{\langle x,\tilde{\mu}_{\tilde{A}}(x),\tilde{\gamma}_{\tilde{A}}(x)\rangle/x\in X\}$ where $\tilde{\mu}_{\tilde{A}}(x)$ and $\tilde{\gamma}_{\tilde{A}}(x)$ are mappings from X to $D[0,1]$ such that $0\le\sup\tilde{\mu}_{\tilde{A}}(x)+\sup\tilde{\gamma}_{\tilde{A}}(x)\le 1\quad\forall\in X.$

The interval $\tilde{\mu}_{\tilde{A}}(x)$ denoted by $\left[\mu^L_{\tilde{A}}(x),\mu^U_{\tilde{A}}(x)\right]$ and $\tilde{\gamma}_{\tilde{A}}(x)$ denoted by $\left[\tilde{\gamma}^L_{\tilde{A}}(x),\tilde{\gamma}^U_{\tilde{A}}(x)\right]$ are the degree of membership and non-membership of x to $\tilde{A}$, respectively where $\tilde{\mu}^L_{\tilde{A}}(x),\tilde{\mu}^U_{\tilde{A}}(x),\tilde{\gamma}^L_{\tilde{A}}(x)$ and $\tilde{\gamma}^U_{\tilde{A}}(x)$ represent the lower and upper bounds of $\tilde{\mu}_{\tilde{A}}(x)$ and $\tilde{\gamma}_{\tilde{A}}(x)$. For any given x, the pair $\left(\tilde{\mu}_{\tilde{A}}(x),\tilde{\gamma}_{\tilde{A}}(x)\right)$ is called an interval intuitionistic fuzzy number (IVIFN) [9]. For convenience an IVIFN is denoted by $\tilde{\alpha} = \left(\left[\mu_\alpha^L, \mu_\alpha^U\right], \left[\gamma_\alpha^L, \gamma_\alpha^U\right]\right)$ where $\left[\mu_\alpha^L, \mu_\alpha^U\right]$ *and* $\left[\gamma_\alpha^L, \gamma_\alpha^U\right] \in D[0,1]$ and $\mu_\alpha^U + \gamma_\alpha^U \leq 1$

Definition 2.3 [9]

Let $\tilde{\alpha}_1 = \left(\left[\mu_{\alpha_1}^L, \mu_{\alpha_1}^U\right], \left[\gamma_{\alpha_1}^L, \gamma_{\alpha_1}^U\right]\right)$ and $\tilde{\alpha}_2 = \left(\left[\mu_{\alpha_2}^L, \mu_{\alpha_2}^U\right], \left[\gamma_{\alpha_2}^L, \gamma_{\alpha_2}^U\right]\right)$ be any two IVIFNs, then some Einstein operations of IVIFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are defined as

1. $\tilde{\alpha}_1^c = \left(\left[\gamma_{\alpha_1}^L, \gamma_{\alpha_1}^U\right], \left[\mu_{\alpha_1}^L, \mu_{\alpha_1}^U\right]\right)$

2. $$\tilde{\alpha}_1 + \tilde{\alpha}_2 = \left(\left[\frac{\mu_{\alpha_1}^L + \mu_{\alpha_2}^L}{1 + \mu_{\alpha_1}^L \mu_{\alpha_2}^L}, \frac{\mu_{\alpha_1}^U + \mu_{\alpha_2}^U}{1 + \mu_{\alpha_1}^U \mu_{\alpha_2}^U}\right], \left[\frac{\gamma_{\alpha_1}^L \gamma_{\alpha_2}^L}{1 + \left(1 - \gamma_{\alpha_1}^L\right)\left(1 - \gamma_{\alpha_2}^L\right)}\right], \left[\frac{\gamma_{\alpha_1}^U \gamma_{\alpha_2}^U}{1 + \left(1 - \gamma_{\alpha_1}^U\right)\left(1 - \gamma_{\alpha_2}^U\right)}\right]\right)$$

3. $$\tilde{\alpha}_1 \times \tilde{\alpha}_2 = \left(\left[\frac{\mu_{\alpha_1}^L \mu_{\alpha_2}^L}{1 + \left(1 - \mu_{\alpha_1}^L\right)\left(1 - \mu_{\alpha_2}^L\right)}\right], \left[\frac{\mu_{\alpha_1}^U \mu_{\alpha_2}^U}{1 + \left(1 - \mu_{\alpha_1}^U\right)\left(1 - \mu_{\alpha_2}^U\right)}\right]\right) + \left(\left[\frac{\gamma_{\alpha_1}^L + \gamma_{\alpha_2}^L}{1 + \gamma_{\alpha_1}^L \gamma_{\alpha_2}^L}, \frac{\gamma_{\alpha_1}^U + \gamma_{\alpha_2}^U}{1 + \gamma_{\alpha_1}^U \gamma_{\alpha_2}^U}\right]\right)$$

4. $$\lambda \tilde{\alpha}_1 = \left(\left[\frac{\left(1 + \mu_{\alpha_1}^L\right)^\lambda - \left(1 - \mu_{\alpha_1}^L\right)^\lambda}{\left(1 + \mu_{\alpha_1}^L\right)^\lambda + \left(1 - \mu_{\alpha_2}^L\right)^\lambda}, \frac{\left(1 + \mu_{\alpha_1}^U\right)^\lambda - \left(1 - \mu_{\alpha_1}^U\right)^\lambda}{\left(1 + \mu_{\alpha_1}^U\right)^\lambda + \left(1 - \mu_{\alpha_2}^U\right)^\lambda}\right]\right)\left(\left[\frac{2\left(\gamma_{\alpha_1}^L\right)^\lambda}{\left(2 - \gamma_{\alpha_1}^L\right)^\lambda + \left(\gamma_{\alpha_1}^L\right)^\lambda}, \frac{2\left(\gamma_{\alpha_1}^U\right)^\lambda}{\left(2 - \gamma_{\alpha_1}^U\right)^\lambda + \left(\gamma_{\alpha_1}^U\right)^\lambda}\right]\right); \lambda > 0$$

5. $$\tilde{\alpha}^\lambda = \left(\left[\frac{2\left(\mu_{\alpha_1}^L\right)^\lambda}{\left(2 - \mu_{\alpha_1}^L\right)^\lambda + \left(\mu_{\alpha_1}^L\right)^\lambda}, \frac{2\left(\mu_{\alpha_1}^U\right)^\lambda}{\left(2 - \mu_{\alpha_1}^U\right)^\lambda + \left(\mu_{\alpha_1}^U\right)^\lambda}\right], \left[\frac{\left(1 + \gamma_{\alpha_1}^L\right)^\lambda - \left(1 - \gamma_{\alpha_1}^L\right)^\lambda}{\left(1 + \gamma_{\alpha_1}^L\right)^\lambda + \left(1 - \gamma_{\alpha_1}^L\right)^\lambda}, \frac{\left(1 + \gamma_{\alpha_1}^U\right)^\lambda - \left(1 - \gamma_{\alpha_1}^U\right)^\lambda}{\left(1 + \gamma_{\alpha_1}^U\right)^\lambda + \left(1 - \gamma_{\alpha_1}^U\right)^\lambda}\right]\right); \lambda > 0$$

3. INTERVAL-VALUED INTUITIONISTIC HESITANT FUZZY SET AND INTERVAL VALUED INTUITIONISTIC HESITANT FUZZY ELEMENTS

The interval valued intuitionistic hesitant fuzzy sets (IVIHFS) allows the membership of an element to be a set of several possible interval-valued intuitionistic fuzzy numbers [10]

Definition 3.1 [10]

Let X be a fixed set, $\tilde{E} = \left\{\left\langle x, h_{\tilde{E}}(x)\right\rangle / x \in X\right\}$ where, $h_{\tilde{E}}(x)$ is a set of some IVIFNs in Ω denoting the possible membership and non-membership degree intervals of the element $x \in X$ to the set $\tilde{E}$. $\tilde{h} = h_{\tilde{E}}(x)$ is called an interval valued intuitionistic hesitant fuzzy element (IVIHFE) and $\tilde{H}$ denotes the set of all IVIHFEs. If $\alpha \in \tilde{h}$ then α is an IVIFN denoted by $\alpha = (\mu_\alpha, \gamma_\alpha) = \left(\left[\mu_\alpha^L, \mu_\alpha^U\right], \left[\gamma_\alpha^L, \gamma_\alpha^U\right]\right)$.

Now we extend the Einstein operation on IVIFNs to IVIHFEs

Definition 3.2

Given three IVIHFEs $\tilde{h}=\left\{\left(\left[\mu_{\alpha}^{L},\mu_{\alpha}^{U}\right],\left[\gamma_{\alpha}^{L},\gamma_{\alpha}^{U}\right]\right)/\tilde{\alpha}\in\tilde{h}\right\}$, $\tilde{h}_1=\left\{\left(\left[\mu_{\alpha_1}^{L},\mu_{\alpha_1}^{U}\right],\left[\gamma_{\alpha_1}^{L},\gamma_{\alpha_1}^{U}\right]\right)/\tilde{\alpha}_1\in\tilde{h}_1\right\}$ and $\tilde{h}_2=\left\{\left(\left[\mu_{\alpha_2}^{L},\mu_{\alpha_2}^{U}\right],\left[\gamma_{\alpha_2}^{L},\gamma_{\alpha_2}^{U}\right]\right)/\tilde{\alpha}_2\in\tilde{h}_2\right\}$ let us define the Einstein operation on them as below

1. $\tilde{h}^{c}=\left\{\alpha^{c}/\alpha\in\tilde{h}\right\}=\left\{\left(\left[\gamma_{\alpha}^{L},\gamma_{\alpha}^{U}\right],\left[\mu_{\alpha}^{L},\mu_{\alpha}^{U}\right]\right)/\tilde{\alpha}\in\tilde{h}\right\}$

2. $\tilde{h}_1\oplus\tilde{h}_2=\left\{\left(\left[\frac{\mu_{\alpha_1}^{L}+\mu_{\alpha_2}^{L}}{1+\mu_{\alpha_1}^{L}\mu_{\alpha_2}^{L}},\frac{\mu_{\alpha_1}^{U}+\mu_{\alpha_2}^{U}}{1+\mu_{\alpha_1}^{U}\mu_{\alpha_2}^{U}}\right],\left[\frac{\gamma_{\alpha_1}^{L}\gamma_{\alpha_2}^{L}}{1+\left(1-\gamma_{\alpha_1}^{L}\right)\left(1-\gamma_{\alpha_2}^{L}\right)}\right],\left[\frac{\gamma_{\alpha_1}^{U}\gamma_{\alpha_2}^{U}}{1+\left(1-\gamma_{\alpha_1}^{U}\right)\left(1-\gamma_{\alpha_2}^{U}\right)}\right]\right)\Big/\alpha_1\in\tilde{h}_1,\alpha_2\in\tilde{h}_2\right\}$

3. $\tilde{h}_1\otimes\tilde{h}_2=\left\{\left(\left[\frac{\mu_{\alpha_1}^{L}\mu_{\alpha_2}^{L}}{1+\left(1-\mu_{\alpha_1}^{L}\right)\left(1-\mu_{\alpha_2}^{L}\right)},\frac{\mu_{\alpha_1}^{U}\mu_{\alpha_2}^{U}}{1+\left(1-\mu_{\alpha_1}^{U}\right)\left(1-\mu_{\alpha_2}^{U}\right)}\right],\left[\frac{\gamma_{\alpha_1}^{L}\gamma_{\alpha_2}^{L}}{1+\gamma_{\alpha_1}^{L}\gamma_{\alpha_2}^{L}},\frac{\gamma_{\alpha_1}^{U}\gamma_{\alpha_2}^{U}}{1+\gamma_{\alpha_1}^{U}\gamma_{\alpha_2}^{U}}\right]\right)\Big/\alpha_1\in\tilde{h}_1,\alpha_2\in\tilde{h}_2\right\}$

4. For $\lambda>0$ $\lambda\tilde{h}_1=\left\{\left(\left[\frac{\left(1+\mu_{\alpha_1}^{L}\right)^{\lambda}-\left(1-\mu_{\alpha_1}^{L}\right)^{\lambda}}{\left(1+\mu_{\alpha_1}^{L}\right)^{\lambda}+\left(1-\mu_{\alpha_2}^{L}\right)^{\lambda}},\frac{\left(1+\mu_{\alpha_1}^{U}\right)^{\lambda}-\left(1-\mu_{\alpha_1}^{U}\right)^{\lambda}}{\left(1+\mu_{\alpha_1}^{U}\right)^{\lambda}+\left(1-\mu_{\alpha_2}^{U}\right)^{\lambda}}\right],\left[\frac{2\left(\gamma_{\alpha_1}^{L}\right)^{\lambda}}{\left(2-\gamma_{\alpha_1}^{L}\right)^{\lambda}+\left(\gamma_{\alpha_1}^{L}\right)^{\lambda}},\frac{2\left(\gamma_{\alpha_1}^{U}\right)^{\lambda}}{\left(2-\gamma_{\alpha_1}^{U}\right)^{\lambda}+\left(\gamma_{\alpha_1}^{U}\right)^{\lambda}}\right]\right)\Big/\begin{array}{c}\alpha_1\in\tilde{h}_1\\ \lambda>0\end{array}\right\}$

5. For $\lambda>0$ $\left(\tilde{h}\right)^{\lambda}=\left\{\left(\left[\frac{2\left(\mu_{\alpha_1}^{L}\right)^{\lambda}}{\left(2-\mu_{\alpha_1}^{L}\right)^{\lambda}+\left(\mu_{\alpha_1}^{L}\right)^{\lambda}},\frac{2\left(\mu_{\alpha_1}^{U}\right)^{\lambda}}{\left(2-\mu_{\alpha_1}^{U}\right)^{\lambda}+\left(\mu_{\alpha_1}^{U}\right)^{\lambda}}\right],\left[\frac{\left(1+\gamma_{\alpha_1}^{L}\right)^{\lambda}-\left(1-\gamma_{\alpha_1}^{L}\right)^{\lambda}}{\left(1+\gamma_{\alpha_1}^{L}\right)^{\lambda}+\left(1-\gamma_{\alpha_1}^{L}\right)^{\lambda}},\frac{\left(1+\gamma_{\alpha_1}^{U}\right)^{\lambda}-\left(1-\gamma_{\alpha_1}^{U}\right)^{\lambda}}{\left(1+\gamma_{\alpha_1}^{U}\right)^{\lambda}+\left(1-\gamma_{\alpha_1}^{U}\right)^{\lambda}}\right]\right)\Big/\alpha\in\tilde{h}\right\}$

Theorem 3.1

Let $\tilde{h},\tilde{h}_1$ *and* $\tilde{h}_2$ be three IVIFHEs and $\lambda\geq 0$. Then $\tilde{h}_1+\tilde{h}_2$, $\tilde{h}_1\times\tilde{h}_2$, $\lambda\tilde{h}$ *and* $\left(\tilde{h}\right)^{\lambda}$ are also IVIHFEs.

Proof

Let $\tilde{h}=\left\{\left(\left[\mu_{\alpha}^{L},\mu_{\alpha}^{U}\right],\left[\gamma_{\alpha}^{L},\gamma_{\alpha}^{U}\right]\right)/\alpha\in\tilde{h}\right\}$, $\tilde{h}_1=\left\{\left(\left[\mu_{\alpha_1}^{L},\mu_{\alpha_1}^{U}\right],\left[\gamma_{\alpha_1}^{L},\gamma_{\alpha_1}^{U}\right]\right)/\alpha_1\in\tilde{h}_1\right\}$

$\tilde{h}_2=\left\{\left(\left[\mu_{\alpha_2}^{L},\mu_{\alpha_2}^{U}\right],\left[\gamma_{\alpha_2}^{L},\gamma_{\alpha_2}^{U}\right]\right)/\alpha_2\in\tilde{h}_2\right\}$ be three IVIHFEs.

Hence by definition, $0\leq\mu_{\alpha}^{L},\mu_{\alpha}^{U},\gamma_{\alpha}^{L},\gamma_{\alpha}^{U},\mu_{\alpha_1}^{L},\mu_{\alpha_1}^{U},\gamma_{\alpha_1}^{L},\gamma_{\alpha_1}^{U},\mu_{\alpha_2}^{L},\mu_{\alpha_2}^{U},\gamma_{\alpha_2}^{L},\gamma_{\alpha_2}^{U}\leq 1$ and $\mu_{\alpha}^{L}+\gamma_{\alpha}^{U}\leq 1,\mu_{\alpha_1}^{L}+\gamma_{\alpha_1}^{U}\leq 1,\mu_{\alpha_2}^{L}+\gamma_{\alpha_2}^{U}\leq 1$.

$0\leq\left(1-\mu_{\alpha_1}^{L}\right)\left(1-\mu_{\alpha_2}^{L}\right)=1-\mu_{\alpha_1}^{L}-\mu_{\alpha_2}^{L}+\mu_{\alpha_1}^{L}\mu_{\alpha_2}^{L}$ and $\mu_{\alpha_1}^{L}+\mu_{\alpha_2}^{L}\leq 1+\mu_{\alpha_1}^{L}\mu_{\alpha_2}^{L}$.

Thus $\frac{\mu_{\alpha_1}^{L}+\mu_{\alpha_2}^{L}}{1+\mu_{\alpha_1}^{L}\mu_{\alpha_2}^{L}}\leq 1$. Obviously $\frac{\mu_{\alpha_1}^{L}+\mu_{\alpha_2}^{L}}{1+\mu_{\alpha_1}^{L}\mu_{\alpha_2}^{L}}\geq 0$. Hence, $0\leq\frac{\mu_{\alpha_1}^{L}+\mu_{\alpha_2}^{L}}{1+\mu_{\alpha_1}^{L}\mu_{\alpha_2}^{L}}\leq 1$. Similarly $0\leq\frac{\mu_{\alpha_1}^{U}+\mu_{\alpha_2}^{U}}{1+\mu_{\alpha_1}^{U}\mu_{\alpha_2}^{U}}\leq 1$.

Since $0\leq\gamma_{\alpha_1}^{L}\leq 1$ and $0\leq\gamma_{\alpha_2}^{L}\leq 1$

$$\gamma^L_{\alpha_1}+\gamma^L_{\alpha_2}\le 2 \Leftrightarrow 0\le 2-\gamma^L_{\alpha_1}-\gamma^L_{\alpha_2} \Leftrightarrow \gamma^L_{\alpha_1}\gamma^L_{\alpha_2}\le 2-\gamma^L_{\alpha_1}-\gamma^L_{\alpha_2}+\gamma^L_{\alpha_1}\gamma^L_{\alpha_2}$$

$$\Leftrightarrow \gamma^L_{\alpha_1}\gamma^L_{\alpha_2}\le 1+\left(1-\gamma^L_{\alpha_1}\right)\left(1-\gamma^L_{\alpha_2}\right) \Leftrightarrow \frac{\gamma^L_{\alpha_1}\gamma^L_{\alpha_2}}{1+\left(1-\gamma^L_{\alpha_1}\right)\left(1-\gamma^L_{\alpha_2}\right)}\le 1$$

Thus $0\le \frac{\gamma^L_{\alpha_1}\gamma^L_{\alpha_2}}{1+\left(1-\gamma^L_{\alpha_1}\right)\left(1-\gamma^L_{\alpha_2}\right)}\le 1$. Similarly, $0\le \frac{\gamma^U_{\alpha_1}\gamma^U_{\alpha_2}}{1+\left(1-\gamma^U_{\alpha_1}\right)\left(1-\gamma^U_{\alpha_2}\right)}\le 1$

$\frac{\mu^U_{\alpha_1}+\gamma^U_{\alpha_2}}{1+\mu^U_{\alpha_1}\mu^U_{\alpha_2}}+\frac{\gamma^U_{\alpha_1}\gamma^U_{\alpha_2}}{1+\left(1-\gamma^U_{\alpha_1}\right)\left(1-\gamma^U_{\alpha_2}\right)}\le \frac{\mu^U_{\alpha_1}+\mu^U_{\alpha_2}}{1+\mu^U_{\alpha_1}\mu^U_{\alpha_2}}+\frac{\left(1-\mu^U_{\alpha_1}\right)\left(1-\mu^U_{\alpha_2}\right)}{1+\mu^U_{\alpha_1}\mu^U_{\alpha_2}}$ since $\mu^U_{\alpha_1}+\gamma^U_{\alpha_1}\le 1$ and $\mu^U_{\alpha_2}+\gamma^U_{\alpha_2}\le 1$

Thus $\frac{\mu^U_{\alpha_1}+\gamma^U_{\alpha_2}}{1+\mu^U_{\alpha_1}\mu^U_{\alpha_2}}+\frac{\gamma^U_{\alpha_1}\gamma^U_{\alpha_2}}{1+\left(1-\gamma^U_{\alpha_1}\right)\left(1-\gamma^U_{\alpha_2}\right)}\le \frac{\mu^U_{\alpha_1}+\mu^U_{\alpha_2}+1-\mu^U_{\alpha_1}-\mu^U_{\alpha_2}+\mu^U_{\alpha_1}\mu^U_{\alpha_2}}{1+\mu^U_{\alpha_1}\mu^U_{\alpha_2}}=1$

Hence $\tilde{h}_1\oplus\tilde{h}_2$ is an IVIHFE.

- **To Prove $\tilde{h}_1\otimes\tilde{h}_2$ is an IVIHFE**

Since $0\le \mu^L_{\alpha_1},\mu^U_{\alpha_1}\le 1$, we have $\mu^L_{\alpha_1}+\mu^L_{\alpha_2}\le 2$

Thus $0\le 2-\mu^L_{\alpha_1}-\mu^L_{\alpha_2} \Leftrightarrow \mu^L_{\alpha_1}\mu^L_{\alpha_2}\le 1+1-\mu^L_{\alpha_1}-\mu^L_{\alpha_2}+\mu^L_{\alpha_1}\mu^L_{\alpha_2}=1+\left(1-\mu^L_{\alpha_1}\right)\left(1-\mu^L_{\alpha_2}\right)$

$\therefore \frac{\mu^L_{\alpha_1}\mu^L_{\alpha_2}}{1+\left(1-\mu^L_{\alpha_1}\right)\left(1-\mu^L_{\alpha_2}\right)}\le 1$. Similarly, it is true for $\mu^U_{\alpha_1},\mu^U_{\alpha_2}$

Also, $0\le \frac{\gamma^L_{\alpha_1}+\gamma^L_{\alpha_2}}{1+\gamma^L_{\alpha_1}\gamma^L_{\alpha_2}},\frac{\gamma^U_{\alpha_1}+\gamma^U_{\alpha_2}}{1+\gamma^U_{\alpha_1}\gamma^U_{\alpha_2}}\le 1$

$\frac{\mu^U_{\alpha_1}\mu^U_{\alpha_2}}{1+\left(1-\mu^U_{\alpha_1}\right)\left(1-\mu^U_{\alpha_2}\right)}+\frac{\gamma^U_{\alpha_1}+\gamma^U_{\alpha_2}}{1+\gamma^U_{\alpha_1}\gamma^U_{\alpha_2}}\le \frac{\left(1-\gamma^U_{\alpha_1}\right)\left(1-\gamma^U_{\alpha_2}\right)}{1+\gamma^U_{\alpha_1}\gamma^U_{\alpha_2}}+\frac{\gamma^U_{\alpha_1}+\gamma^U_{\alpha_2}}{1+\gamma^U_{\alpha_1}\gamma^U_{\alpha_2}}$ since $\mu^U_{\alpha_1}+\gamma^U_{\alpha_1}\le 1$ and

$=\frac{1-\gamma^U_{\alpha_1}-\gamma^U_{\alpha_2}+\gamma^U_{\alpha_1}\gamma^U_{\alpha_2}+\gamma^U_{\alpha_1}+\gamma^U_{\alpha_2}}{1+\gamma^U_{\alpha_1}\gamma^U_{\alpha_2}}=1$

Thus, $\tilde{h}_1\otimes\tilde{h}_2$ is an IVIHFE.

- **To Prove $\lambda\tilde{h}_1$ is an IVIHFE**

$$\left(1+\mu^L_{\alpha_1}\right)^{\lambda}-\left(1-\mu^L_{\alpha_1}\right)^{\lambda}\le\left(1+\mu^L_{\alpha_1}\right)^{\lambda}+\left(1-\mu^L_{\alpha_1}\right)^{\lambda}$$

Thus $\dfrac{\left(1+\mu_{\alpha_1}^L\right)^\lambda-\left(1-\mu_{\alpha_1}^L\right)^\lambda}{\left(1+\mu_{\alpha_1}^L\right)^\lambda+\left(1-\mu_{\alpha_1}^L\right)^\lambda}\le 1$. Similar results hold for $\mu_{\alpha_1}^U$ also.

As $0\le\gamma_{\alpha_1}^L\le 1,\quad \lambda>0,\quad \textit{we have}\ 2\left(\gamma_{\alpha_1}^L\right)^\lambda\le\left(2-\gamma_{\alpha_1}^L\right)^\lambda+\left(\gamma_{\alpha_1}^L\right)^\lambda$

$$\therefore \frac{2\left(\gamma_{\alpha_1}^L\right)^\lambda}{\left(2-\gamma_{\alpha_1}^L\right)^\lambda+\left(\gamma_{\alpha_1}^L\right)^\lambda}\le 1$$

Similar results is true for $\gamma_{\alpha_1}^U$ also.

As $\mu_{\alpha_1}^L+\gamma_{\alpha_1}^L\le 1\ \textit{and}\ \lambda>0$

$$\frac{\left(1+\mu_{\alpha_1}^U\right)^\lambda-\left(1-\mu_{\alpha_1}^U\right)^\lambda}{\left(1+\mu_{\alpha_1}^U\right)^\lambda+\left(1-\mu_{\alpha_1}^U\right)^\lambda}+\frac{2\left(\gamma_{\alpha_1}^U\right)^\lambda}{\left(2-\gamma_{\alpha_1}^U\right)^\lambda+\left(\gamma_{\alpha_1}^U\right)^\lambda}\le\frac{\left(1+\mu_{\alpha_1}^U\right)^\lambda-\left(1-\mu_{\alpha_1}^U\right)^\lambda}{\left(1+\mu_{\alpha_1}^U\right)^\lambda+\left(1-\mu_{\alpha_1}^U\right)^\lambda}+\frac{2\left(1-\mu_{\alpha_1}^U\right)^\lambda}{\left(1+\mu_{\alpha_1}^U\right)^\lambda+\left(1-\mu_{\alpha_1}^U\right)^\lambda}=1$$

Thus $\lambda\tilde{h}_1$ is an IVIHFE

Similarly we can prove that $\tilde{h}_1^{\ \lambda}$ is an IVIHFE.

Theorem 3.2: Let $\tilde{h},\tilde{h}_1\ \textit{and}\ \tilde{h}_2$ be three IVIHFEs and $\lambda,\lambda_1,\lambda_2>0$ Then

- $\tilde{h}_1\oplus\tilde{h}_2=\tilde{h}_2\oplus\tilde{h}_1$
- $\tilde{h}_1\otimes\tilde{h}_2=\tilde{h}_2\otimes\tilde{h}_1$
- $\lambda(\tilde{h}_1\oplus\tilde{h}_2)=\lambda(\tilde{h}_2\oplus\tilde{h}_1)$
- $(\tilde{h}_1\otimes\tilde{h}_2)^\lambda=\tilde{h}_2^{\ \lambda}\otimes\tilde{h}_1^{\ \lambda}$
- $\left(\lambda_1+\lambda_2\right)\tilde{h}=\lambda_1\tilde{h}\oplus\lambda_2\tilde{h}$
- $\tilde{h}^{\lambda_1}\otimes\tilde{h}^{\lambda_2}=\left(\tilde{h}\right)^{\lambda_1+\lambda_2}$

Proof

- $\tilde{h}_1\oplus\tilde{h}_2=\left\{\left[\dfrac{\mu_{\alpha_1}^L+\mu_{\alpha_2}^L}{1+\mu_{\alpha_1}^L\mu_{\alpha_2}^L},\dfrac{\mu_{\alpha_1}^U+\mu_{\alpha_2}^U}{1+\mu_{\alpha_1}^U\mu_{\alpha_2}^U}\right]\left[\dfrac{\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L}{1+\left(1-\gamma_{\alpha_1}^L\right)\left(1-\gamma_{\alpha_2}^L\right)},\dfrac{\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U}{1+\left(1-\gamma_{\alpha_1}^U\right)\left(1-\gamma_{\alpha_2}^U\right)}\right]\Big/\alpha_1\in\tilde{h}_1,\alpha_2\in\tilde{h}_2\right\}$

$$= \left\{ \left[\frac{\mu_{\alpha_2}^L + \mu_{\alpha_1}^L}{1+\mu_{\alpha_2}^L \mu_{\alpha_1}^L}, \frac{\mu_{\alpha_2}^U + \mu_{\alpha_1}^U}{1+\mu_{\alpha_2}^U \mu_{\alpha_1}^U} \right], \left[\frac{\gamma_{\alpha_2}^L \gamma_{\alpha_1}^L}{1+\left(1-\gamma_{\alpha_2}^L\right)\left(1-\gamma_{\alpha_1}^L\right)}, \frac{\gamma_{\alpha_2}^U \gamma_{\alpha_1}^U}{1+\left(1-\gamma_{\alpha_2}^U\right)\left(1-\gamma_{\alpha_1}^U\right)} \right] \Big/ \alpha_2 \in \tilde{h}_2, \alpha_1 \in \tilde{h}_2 \right\}$$

$$= \tilde{h}_2 \oplus \tilde{h}_1$$

- $\tilde{h}_1 \otimes \tilde{h}_2 = \tilde{h}_2 \otimes \tilde{h}_1$ is obvious as addition and multiplication are commutative.

- $$\tilde{h}_1 \oplus \tilde{h}_2 = \left\{ \left(\left[\frac{\mu_{\alpha_1}^L + \mu_{\alpha_2}^L}{1+\mu_{\alpha_1}^L \mu_{\alpha_2}^L}, \frac{\mu_{\alpha_1}^U + \mu_{\alpha_2}^U}{1+\mu_{\alpha_1}^U \mu_{\alpha_2}^U} \right], \left[\frac{\gamma_{\alpha_1}^L \gamma_{\alpha_2}^L}{1+\left(1-\gamma_{\alpha_1}^L\right)\left(1-\gamma_{\alpha_2}^L\right)}, \frac{\gamma_{\alpha_1}^U \gamma_{\alpha_2}^U}{1+\left(1-\gamma_{\alpha_1}^U\right)\left(1-\gamma_{\alpha_2}^U\right)} \right] \right) \Big/ \alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2 \right\}$$

$$\lambda\left(\tilde{h}_1 \oplus \tilde{h}_2\right) = \left\{ \left(\left[\frac{\left(1+\frac{\mu_{\alpha_1}^L + \mu_{\alpha_2}^L}{1+\mu_{\alpha_1}^L \mu_{\alpha_2}^L}\right)^\lambda - \left(1-\frac{\mu_{\alpha_1}^L + \mu_{\alpha_2}^L}{1+\mu_{\alpha_1}^L \mu_{\alpha_2}^L}\right)^\lambda}{\left(1+\frac{\mu_{\alpha_1}^L + \mu_{\alpha_2}^L}{1+\mu_{\alpha_1}^L \mu_{\alpha_2}^L}\right)^\lambda + \left(1-\frac{\mu_{\alpha_1}^L + \mu_{\alpha_2}^L}{1+\mu_{\alpha_1}^L \mu_{\alpha_2}^L}\right)^\lambda}, \frac{\left(1+\frac{\mu_{\alpha_1}^U + \mu_{\alpha_2}^U}{1+\mu_{\alpha_1}^U \mu_{\alpha_2}^U}\right)^\lambda - \left(1-\frac{\mu_{\alpha_1}^U + \mu_{\alpha_2}^U}{1+\mu_{\alpha_1}^U \mu_{\alpha_2}^U}\right)^\lambda}{\left(1+\frac{\mu_{\alpha_1}^U + \mu_{\alpha_2}^U}{1+\mu_{\alpha_1}^U \mu_{\alpha_2}^U}\right)^\lambda + \left(1-\frac{\mu_{\alpha_1}^U + \mu_{\alpha_2}^U}{1+\mu_{\alpha_1}^U \mu_{\alpha_2}^U}\right)^\lambda} \right], \right.\right.$$

$$\left.\left. \left[\frac{2\left(\frac{\gamma_{\alpha_1}^L \gamma_{\alpha_2}^L}{1+\left(1-\gamma_{\alpha_1}^L\right)\left(1-\gamma_{\alpha_2}^L\right)}\right)^\lambda}{\left(2-\frac{\gamma_{\alpha_1}^L \gamma_{\alpha_2}^L}{1+\left(1-\gamma_{\alpha_1}^L\right)\left(1-\gamma_{\alpha_2}^L\right)}\right)^\lambda + \left(\frac{\gamma_{\alpha_1}^L \gamma_{\alpha_2}^L}{1+\left(1-\gamma_{\alpha_1}^L\right)\left(1-\gamma_{\alpha_2}^L\right)}\right)^\lambda} + \frac{2\left(\frac{\gamma_{\alpha_1}^U \gamma_{\alpha_2}^U}{1+\left(1-\gamma_{\alpha_1}^U\right)\left(1-\gamma_{\alpha_2}^U\right)}\right)^\lambda}{\left(2-\frac{\gamma_{\alpha_1}^U \gamma_{\alpha_2}^U}{1+\left(1-\gamma_{\alpha_1}^U\right)\left(1-\gamma_{\alpha_2}^U\right)}\right)^\lambda + \left(\frac{\gamma_{\alpha_1}^U \gamma_{\alpha_2}^U}{1+\left(1-\gamma_{\alpha_1}^U\right)\left(1-\gamma_{\alpha_2}^U\right)}\right)^\lambda} \right] \right) \Big/ \alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2 \right\}$$

$$= \left\{ \left[\frac{\left(1+\mu_{\alpha_1}^L\right)^\lambda \left(1+\mu_{\alpha_2}^L\right)^\lambda - \left(1-\mu_{\alpha_1}^L\right)^\lambda \left(1-\mu_{\alpha_2}^L\right)^\lambda}{\left(1+\mu_{\alpha_1}^L\right)^\lambda \left(1+\mu_{\alpha_2}^L\right)^\lambda + \left(1-\mu_{\alpha_1}^L\right)^\lambda \left(1-\mu_{\alpha_2}^L\right)^\lambda}, \frac{\left(1+\mu_{\alpha_1}^U\right)^\lambda \left(1+\mu_{\alpha_2}^U\right)^\lambda - \left(1-\mu_{\alpha_1}^L\right)^\lambda \left(1-\mu_{\alpha_2}^L\right)^\lambda}{\left(1+\mu_{\alpha_1}^U\right)^\lambda \left(1+\mu_{\alpha_2}^U\right)^\lambda + \left(1-\mu_{\alpha_1}^U\right)^\lambda \left(1-\mu_{\alpha_2}^U\right)^\lambda} \right], \right.$$

$$\left. \left[\frac{2\gamma_{\alpha_1}^{L^\lambda} \gamma_{\alpha_2}^{L^\lambda}}{\left(2-\gamma_{\alpha_1}^L\right)^\lambda \left(2-\gamma_{\alpha_2}^L\right)^\lambda + \gamma_{\alpha_1}^{L^\lambda} \gamma_{\alpha_2}^{L^\lambda}}, \frac{2\gamma_{\alpha_1}^{U^\lambda} \gamma_{\alpha_2}^{U^\lambda}}{\left(2-\gamma_{\alpha_1}^U\right)^\lambda \left(2-\gamma_{\alpha_2}^U\right)^\lambda + \gamma_{\alpha_1}^{U^\lambda} \gamma_{\alpha_2}^{U^\lambda}} \right] \Big/ \alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2 \right\} \rightarrow (1)$$

$$\lambda\tilde{h}_1 = \left\{ \left(\left[\frac{\left(1+\mu_{\alpha_1}^L\right)^\lambda - \left(1-\mu_{\alpha_1}^L\right)^\lambda}{\left(1+\mu_{\alpha_1}^L\right)^\lambda + \left(1-\mu_{\alpha_1}^L\right)^\lambda}, \frac{\left(1+\mu_{\alpha_1}^U\right)^\lambda - \left(1-\mu_{\alpha_1}^U\right)^\lambda}{\left(1+\mu_{\alpha_1}^U\right)^\lambda + \left(1-\mu_{\alpha_1}^U\right)^\lambda} \right], \left[\frac{2\left(\gamma_{\alpha_1}^L\right)^\lambda}{\left(2-\gamma_{\alpha_1}^L\right)^\lambda + \left(\gamma_{\alpha_1}^L\right)^\lambda}, \frac{2\left(\gamma_{\alpha_1}^U\right)^\lambda}{\left(2-\gamma_{\alpha_1}^U\right)^\lambda + \left(\gamma_{\alpha_1}^U\right)^\lambda} \right] \right) / \alpha_1 \in \tilde{h}_1 \right\}$$

$$\lambda\tilde{h}_2 = \left\{ \left(\left[\frac{\left(1+\mu_{\alpha_2}^L\right)^\lambda - \left(1-\mu_{\alpha_2}^L\right)^\lambda}{\left(1+\mu_{\alpha_2}^L\right)^\lambda + \left(1-\mu_{\alpha_2}^L\right)^\lambda}, \frac{\left(1+\mu_{\alpha_2}^U\right)^\lambda - \left(1-\mu_{\alpha_2}^U\right)^\lambda}{\left(1+\mu_{\alpha_2}^U\right)^\lambda + \left(1-\mu_{\alpha_2}^U\right)^\lambda} \right], \left[\frac{2\left(\gamma_{\alpha_2}^L\right)^\lambda}{\left(2-\gamma_{\alpha_2}^L\right)^\lambda + \left(\gamma_{\alpha_2}^L\right)^\lambda}, \frac{2\left(\gamma_{\alpha_2}^U\right)^\lambda}{\left(2-\gamma_{\alpha_2}^U\right)^\lambda + \left(\gamma_{\alpha_2}^U\right)^\lambda} \right] \right) / \alpha_2 \in \tilde{h}_2 \right\}$$

$$\lambda\tilde{h}_1 = \left\{ \left(\left[\frac{A_1^L - B_1^L}{A_1^L + B_1^L}, \frac{A_1^U - B_1^U}{A_1^U + B_1^U} \right], \left[\frac{2E_1^L}{G_1^L + E_1^L}, \frac{2E_1^U}{G_1^U + E_1^U} \right] \right) \right\}$$

$$\lambda\tilde{h}_2=\left\{\left(\left[\frac{A_2^L-B_2^L}{A_2^L+B_2^L},\frac{A_2^U-B_2^U}{A_2^L+B_2^L}\right],\left[\frac{2E_2^L}{G_2^L+E_2^L},\frac{2E_2^U}{G_2^U+E_2^U}\right]\right)\right\}$$

where $A_1^L=\left(1+\mu_{\alpha_1}^L\right)^\lambda, B_1^L=\left(1-\mu_{\alpha_1}^L\right)^\lambda, A_1^U=\left(1+\mu_{\alpha_1}^U\right)^\lambda, B_1^U=\left(1-\mu_{\alpha_1}^U\right)^\lambda$

$A_2^L=\left(1+\mu_{\alpha_2}^L\right)^\lambda, B_2^L=\left(1-\mu_{\alpha_2}^L\right)^\lambda, A_2^U=\left(1+\mu_{\alpha_2}^U\right)^\lambda, B_2^U=\left(1-\mu_{\alpha_2}^U\right)^\lambda$

$E_1^L=\left(\gamma_{\alpha_1}^L\right)^\lambda, G_1^L=\left(2-\gamma_{\alpha_1}^L\right)^\lambda, E_1^U=\left(\gamma_{\alpha_1}^U\right)^\lambda, G_1^U=\left(2-\gamma_{\alpha_1}^U\right)^\lambda$

$E_2^L=\left(\gamma_{\alpha_2}^L\right)^\lambda, G_2^L=\left(2-\gamma_{\alpha_2}^L\right)^\lambda, E_2^U=\left(\gamma_{\alpha_2}^U\right)^\lambda, G_2^U=\left(2-\gamma_{\alpha_2}^U\right)^\lambda$

$$\lambda\tilde{h}_1\oplus\lambda\tilde{h}_2=\left\{\left(\left[\frac{\frac{A_1^L-B_1^L}{A_1^L+B_1^L}+\frac{A_2^L-B_2^L}{A_2^L+B_2^L}}{1+\frac{A_1^L-B_1^L}{A_1^L+B_1^L}+\frac{A_2^L-B_2^L}{A_2^L+B_2^L}},\frac{\frac{A_1^U-B_1^U}{A_1^U+B_1^U}+\frac{A_2^U-B_2^U}{A_2^U+B_2^U}}{1+\frac{A_1^U-B_1^U}{A_1^U+B_1^U}+\frac{A_2^U-B_2^U}{A_2^U+B_2^U}}\right],\left[\frac{2E_1^L}{G_1^L+E_1^L}\times\frac{2E_2^L}{G_2^L+E_2^L},\frac{2E_1^U}{G_1^U+E_1^U}\times\frac{2E_2^U}{G_2^U+E_2^U}\right]\right)\right\}$$

$$=\left\{\left(\left[\frac{A_1^LA_2^L-B_1^LB_2^L}{A_1^LA_2^L+B_1^LB_2^L},\frac{A_1^UA_2^U-B_1^UB_2^U}{A_1^UA_2^U+B_1^UB_2^U}\right],\left[\frac{2E_1^LE_2^L}{G_1^LG_2^L+E_1^LE_2^L},\frac{2E_1^UE_2^U}{G_1^UG_2^U+E_1^UE_2^U}\right]\right)\right\}$$

$$=\left\{\left(\left[\frac{\left(1+\mu_{\alpha_1}^L\right)^\lambda\left(1+\mu_{\alpha_2}^L\right)^\lambda-\left(1-\mu_{\alpha_1}^L\right)^\lambda\left(1-\mu_{\alpha_2}^L\right)^\lambda}{\left(1+\mu_{\alpha_1}^L\right)^\lambda\left(1+\mu_{\alpha_2}^L\right)^\lambda+\left(1-\mu_{\alpha_1}^L\right)^\lambda\left(1-\mu_{\alpha_2}^L\right)^\lambda},\frac{\left(1+\mu_{\alpha_1}^U\right)^\lambda\left(1+\mu_{\alpha_2}^U\right)^\lambda-\left(1-\mu_{\alpha_1}^U\right)^\lambda\left(1-\mu_{\alpha_2}^U\right)^\lambda}{\left(1+\mu_{\alpha_1}^U\right)^\lambda\left(1+\mu_{\alpha_2}^U\right)^\lambda+\left(1-\mu_{\alpha_1}^U\right)^\lambda\left(1-\mu_{\alpha_2}^U\right)^\lambda}\right],\right.\right.$$

$$\left.\left.\left[\frac{2\left(\gamma_{\alpha_1}^L\right)^\lambda\left(\gamma_{\alpha_2}^L\right)^\lambda}{\left(2-\gamma_{\alpha_1}^L\right)^\lambda\left(2-\gamma_{\alpha_2}^L\right)^\lambda+\left(\gamma_{\alpha_1}^L\right)^\lambda\left(\gamma_{\alpha_2}^L\right)^\lambda},\frac{2\left(\gamma_{\alpha_1}^U\right)^\lambda\left(\gamma_{\alpha_2}^U\right)^\lambda}{\left(2-\gamma_{\alpha_1}^L\right)^\lambda\left(2-\gamma_{\alpha_2}^U\right)^\lambda+\left(\gamma_{\alpha_1}^U\right)^\lambda\left(\gamma_{\alpha_2}^U\right)^\lambda}\right]\right)\middle/\alpha_1\in\tilde{h}_1,\alpha_2\in\tilde{h}_2\right\}\rightarrow(2)$$

From (1) and (2), $\lambda\left(\tilde{h}_1\oplus\tilde{h}_2\right)=\lambda\tilde{h}_1\oplus\lambda\tilde{h}_2$

- $\left(\tilde{h}_1\otimes\tilde{h}_2\right)^\lambda=\tilde{h}_1^{\ \lambda}\otimes\tilde{h}_2^{\ \lambda}$

$$\tilde{h}_1^{\ \lambda}=\left\{\left(\left[\frac{2\left(\mu_{\alpha_1}^L\right)^\lambda}{\left(2-\mu_{\alpha_1}^L\right)^\lambda+\left(\mu_{\alpha_1}^L\right)^\lambda},\frac{2\left(\mu_{\alpha_1}^U\right)^\lambda}{\left(2-\mu_{\alpha_1}^U\right)^\lambda+\left(\mu_{\alpha_1}^U\right)^\lambda}\right],\right.\right.$$

$$\left.\left.\left[\frac{\left(1+\gamma_{\alpha_1}^L\right)^\lambda-\left(1-\gamma_{\alpha_1}^L\right)^\lambda}{\left(1+\gamma_{\alpha_1}^L\right)^\lambda+\left(1-\gamma_{\alpha_1}^L\right)^\lambda},\frac{\left(1+\gamma_{\alpha_1}^U\right)^\lambda-\left(1-\gamma_{\alpha_1}^U\right)^\lambda}{\left(1+\gamma_{\alpha_1}^U\right)^\lambda+\left(1-\gamma_{\alpha_1}^U\right)^\lambda}\right]\right)\middle/\alpha_1\in\tilde{h}_1,\lambda>0\right\}$$

$$\tilde{h}_2^{\ \lambda} = \left\{ \left(\left[\frac{2\left(\mu_{\alpha_2}^L\right)^\lambda}{\left(2-\mu_{\alpha_2}^L\right)^\lambda + \left(\mu_{\alpha_2}^L\right)^\lambda}, \frac{2\left(\mu_{\alpha_2}^U\right)^\lambda}{\left(2-\mu_{\alpha_2}^U\right)^\lambda + \left(\mu_{\alpha_2}^U\right)^\lambda} \right], \right. \right.$$

$$\left. \left. \left[\frac{\left(1+\gamma_{\alpha_2}^L\right)^\lambda - \left(1-\gamma_{\alpha_2}^L\right)^\lambda}{\left(1+\gamma_{\alpha_2}^L\right)^\lambda + \left(1-\gamma_{\alpha_2}^L\right)^\lambda}, \frac{\left(1+\gamma_{\alpha_2}^U\right)^\lambda - \left(1-\gamma_{\alpha_2}^U\right)^\lambda}{\left(1+\gamma_{\alpha_2}^U\right)^\lambda + \left(1-\gamma_{\alpha_2}^U\right)^\lambda} \right] \right) \middle/ \alpha_2 \in \tilde{h}_2, \lambda > 0 \right\}$$

Let $\tilde{h}_i^{\ \lambda} = \left\{ \left(\left[\frac{2A_i^L}{B_i^L + A_i^L}, \frac{2A_i^U}{B_i^U + A_i^U} \right], \left[\frac{C_i^L - D_i^L}{C_i^L + D_i^L}, \frac{C_i^U - D_i^U}{C_i^U + D_i^U} \right] \right) \right\}$ where

$$A_i^L = \left(\mu_{\alpha_i}^L\right)^\lambda, A_i^U = \left(\mu_{\alpha_i}^U\right)^\lambda, B_i^L = \left(2-\mu_{\alpha_i}^L\right)^\lambda, B_i^U = \left(2-\mu_{\alpha_i}^U\right)^\lambda$$

$$C_i^L = \left(1+\gamma_{\alpha_i}^L\right)^\lambda, C_i^U = \left(1+\gamma_{\alpha_i}^U\right)^\lambda, D_i^L = \left(1-\gamma_{\alpha_i}^L\right)^\lambda, D_i^U = \left(1-\gamma_{\alpha_i}^U\right)^\lambda \ \ for \ \ i = 1,2$$

$$\tilde{h}_1^{\ \lambda} \otimes \tilde{h}_2^{\ \lambda} = \left\{ \left(\left[\frac{\frac{2A_1^L}{B_1^L + A_1^L} \times \frac{2A_2^L}{B_2^L + A_2^L}}{1 + \left(1 - \frac{2A_1^L}{B_1^L + A_1^L}\right)\left(1 - \frac{2A_2^L}{B_2^L + A_2^L}\right)}, \frac{\frac{2A_1^U}{B_1^U + A_1^L} \times \frac{2A_2^U}{B_2^L + A_2^L}}{1 + \left(1 - \frac{2A_1^U}{B_1^U + A_1^U}\right)\left(1 - \frac{2A_2^U}{B_2^U + A_2^U}\right)} \right] \right. \right.$$

$$\left. \left. \left[\frac{\frac{C_1^L - D_1^L}{C_1^L + D_1^L} + \frac{C_2^L - D_2^L}{C_2^L + D_2^L}}{1 + \left(\frac{C_1^L - D_1^L}{C_1^L + D_1^L} \times \frac{C_2^L - D_2^L}{C_2^L + D_2^L} \right)}, \frac{\frac{C_1^U - D_1^U}{C_1^U + D_1^U} + \frac{C_2^U - D_2^U}{C_2^U + D_2^U}}{1 + \left(\frac{C_1^U - D_1^U}{C_1^U + D_1^U} \times \frac{C_2^U - D_2^U}{C_2^U + D_2^U} \right)} \right] \right) \right\}$$

$$= \left\{ \left(\left[\begin{matrix} 4A_1^L A_2^L \Big/ \left(\left(B_1^L + A_1^L\right)\left(B_2^L + A_2^L\right) + \left(B_1^L - A_1^L\right)\left(B_2^L - A_2^L\right) \right), \\ 4A_1^U A_2^U \Big/ \left(\left(B_1^U + A_1^U\right)\left(B_2^U + A_2^U\right) + \left(B_1^U - A_1^U\right)\left(B_2^U - A_2^U\right) \right) \end{matrix} \right], \left[\frac{C_1^L C_2^L - D_1^L D_2^L}{C_1^L C_2^L + D_1^L D_2^L}, \frac{C_1^U C_2^U - D_1^U D_2^U}{C_1^U C_2^U + D_1^U D_2^U} \right] \right) \right\}$$

$$= \left\{ \left[\frac{2A_1^L A_2^L}{B_1^L B_2^L + A_1^L A_2^L}, \frac{2A_1^U A_2^U}{B_1^U B_2^U + A_1^U A_2^U} \right], \left[\frac{C_1^L C_2^L - D_1^L D_2^L}{C_1^L C_2^L + D_1^L D_2^L}, \frac{C_1^U C_2^U - D_1^U D_2^U}{C_1^U C_2^U + D_1^U D_2^U} \right] \right\}$$

$$= \left\{ \left(\left[\frac{2\mu_{\alpha_1}^{L^\lambda} \gamma_{\alpha_2}^{L^\lambda}}{\left(2-\mu_{\alpha_1}^L\right)^\lambda \left(2-\mu_{\alpha_2}^L\right)^\lambda + \left(\mu_{\alpha_1}^L\right)^\lambda \left(\gamma_{\alpha_2}^L\right)^\lambda}, \frac{2\mu_{\alpha_1}^{U^\lambda} \gamma_{\alpha_2}^{U^\lambda}}{\left(2-\mu_{\alpha_1}^U\right)^\lambda \left(2-\mu_{\alpha_2}^U\right)^\lambda + \left(\mu_{\alpha_1}^U\right)^\lambda \left(\gamma_{\alpha_2}^U\right)^\lambda} \right], \right. \right.$$

$$\left[\frac{\left(1+\gamma_{\alpha_1}^L\right)^\lambda\left(1+\gamma_{\alpha_2}^L\right)^\lambda-\left(1-\gamma_{\alpha_1}^L\right)^\lambda\left(1-\gamma_{\alpha_2}^L\right)^\lambda}{\left(1+\gamma_{\alpha_1}^L\right)^\lambda\left(1+\gamma_{\alpha_2}^L\right)^\lambda+\left(1-\gamma_{\alpha_1}^L\right)^\lambda\left(1-\gamma_{\alpha_2}^L\right)^\lambda},\frac{\left(1+\gamma_{\alpha_1}^U\right)^\lambda\left(1+\gamma_{\alpha_2}^U\right)^\lambda-\left(1-\gamma_{\alpha_1}^U\right)^\lambda\left(1-\gamma_{\alpha_2}^U\right)^\lambda}{\left(1+\gamma_{\alpha_1}^U\right)^\lambda\left(1+\gamma_{\alpha_2}^U\right)^\lambda+\left(1-\gamma_{\alpha_1}^U\right)^\lambda\left(1-\gamma_{\alpha_2}^U\right)^\lambda}\right]\Bigg)\Bigg/\alpha_1\in\tilde{h}_1,\alpha_2\in\tilde{h}_2\Bigg\}\rightarrow(3)$$

$$\left(h_1\otimes h_2\right)=\left\{\left(\left[\frac{\mu_{\alpha_1}^L\mu_{\alpha_2}^L}{1+\left(1-\mu_{\alpha_1}^L\right)\left(1-\mu_{\alpha_2}^L\right)},\frac{\mu_{\alpha_1}^U\mu_{\alpha_2}^U}{1+\left(1-\mu_{\alpha_1}^U\right)\left(1-\mu_{\alpha_2}^U\right)}\right]\left[\frac{\gamma_{\alpha_1}^L+\gamma_{\alpha_2}^L}{1+\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U},\frac{\gamma_{\alpha_1}^U+\gamma_{\alpha_2}^U}{1+\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U}\right]\right)\Bigg/\alpha_1\in\tilde{h}_1,\alpha_2\in\tilde{h}_2\right\}$$

$$\left(h_1\otimes h_2\right)^\lambda=\left\{\left(\left[\frac{\dfrac{2\left(\mu_{\alpha_1}^L\mu_{\alpha_2}^L\right)^\lambda}{\left(1+\left(1-\mu_{\alpha_1}^L\right)\left(1-\mu_{\alpha_2}^L\right)\right)^\lambda}}{\left(2-\left(\dfrac{\mu_{\alpha_1}^L\mu_{\alpha_2}^L}{\left(1+\left(1-\mu_{\alpha_1}^L\right)\left(1-\mu_{\alpha_2}^L\right)\right)^\lambda}\right)\right)^\lambda+\dfrac{\mu_{\alpha_1}^{L^\lambda}\mu_{\alpha_2}^{L^\lambda}}{\left(1+\left(1-\mu_{\alpha_1}^L\right)\left(1-\mu_{\alpha_2}^L\right)\right)^\lambda}},\frac{\dfrac{2\left(\mu_{\alpha_1}^U\mu_{\alpha_2}^U\right)^\lambda}{\left(1+\left(1-\mu_{\alpha_1}^U\right)\left(1-\mu_{\alpha_2}^U\right)\right)^\lambda}}{\left(2-\left(\dfrac{\mu_{\alpha_1}^U\mu_{\alpha_2}^U}{\left(1+\left(1-\mu_{\alpha_1}^U\right)\left(1-\mu_{\alpha_2}^U\right)\right)^\lambda}\right)\right)^\lambda+\dfrac{\mu_{\alpha_1}^{U^\lambda}\mu_{\alpha_2}^{U^\lambda}}{\left(1+\left(1-\mu_{\alpha_1}^U\right)\left(1-\mu_{\alpha_2}^U\right)\right)^\lambda}}\right],\right.\right.$$

$$\left[\frac{\left(1+\dfrac{\gamma_{\alpha_1}^L+\gamma_{\alpha_2}^L}{1+\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L}\right)^\lambda-\left(1-\dfrac{\gamma_{\alpha_1}^L+\gamma_{\alpha_2}^L}{1+\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L}\right)^\lambda}{\left(1+\dfrac{\gamma_{\alpha_1}^L+\gamma_{\alpha_2}^L}{1+\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L}\right)^\lambda+\left(1-\dfrac{\gamma_{\alpha_1}^L+\gamma_{\alpha_2}^L}{1+\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L}\right)^\lambda},\frac{\left(1+\dfrac{\gamma_{\alpha_1}^U+\gamma_{\alpha_2}^U}{1+\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U}\right)^\lambda-\left(1-\dfrac{\gamma_{\alpha_1}^U+\gamma_{\alpha_2}^U}{1+\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U}\right)^\lambda}{\left(1+\dfrac{\gamma_{\alpha_1}^U+\gamma_{\alpha_2}^U}{1+\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U}\right)^\lambda+\left(1-\dfrac{\gamma_{\alpha_1}^U+\gamma_{\alpha_2}^U}{1+\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U}\right)^\lambda}\right]\Bigg)\Bigg/\alpha_1\in\tilde{h}_1,\alpha_2\in\tilde{h}_2\Bigg\}$$

$$=\left\{\left(\left[\frac{2\mu_{\alpha_1}^{L\,\lambda}\mu_{\alpha_2}^{L\,\lambda}}{2\left(1+\left(1-\mu_{\alpha_1}^L\right)\left(1-\mu_{\alpha_2}^L\right)\right)^\lambda},\frac{2\mu_{\alpha_1}^U\mu_{\alpha_2}^U}{2\left(1+\left(1-\mu_{\alpha_1}^U\right)\left(1-\mu_{\alpha_2}^U\right)\right)^\lambda}\right],\right.\right.$$

$$\left[\frac{\left(1+\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L+\gamma_{\alpha_1}^L+\gamma_{\alpha_2}^L\right)^\lambda-\left(1+\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L-\gamma_{\alpha_1}^L-\gamma_{\alpha_2}^L\right)^\lambda}{\left(1+\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L+\gamma_{\alpha_1}^L+\gamma_{\alpha_2}^L\right)^\lambda+\left(1+\gamma_{\alpha_1}^L\gamma_{\alpha_2}^L-\gamma_{\alpha_1}^L-\gamma_{\alpha_2}^L\right)^\lambda},\right.$$

$$\left.\frac{\left(1+\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U+\gamma_{\alpha_1}^U+\gamma_{\alpha_2}^U\right)^\lambda-\left(1+\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U-\gamma_{\alpha_1}^U-\gamma_{\alpha_2}^U\right)^\lambda}{\left(1+\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U+\gamma_{\alpha_1}^U+\gamma_{\alpha_2}^U\right)^\lambda+\left(1+\gamma_{\alpha_1}^U\gamma_{\alpha_2}^U-\gamma_{\alpha_1}^U-\gamma_{\alpha_2}^U\right)^\lambda}\right]\Bigg)\Bigg/\alpha_1\in\tilde{h}_1,\alpha_2\in\tilde{h}_2\Bigg\}$$

$$=\left\{\left(\left[\frac{2\mu_{\alpha_1}^{L^\lambda}\mu_{\alpha_2}^{L^\lambda}}{\left(2-\mu_{\alpha_1}^L\right)^\lambda\left(2-\mu_{\alpha_2}^L\right)^\lambda+\mu_{\alpha_1}^{L^\lambda}\mu_{\alpha_2}^{L^\lambda}},\frac{2\mu_{\alpha_1}^{U^\lambda}\mu_{\alpha_2}^{U^\lambda}}{\left(2-\mu_{\alpha_1}^U\right)^\lambda\left(2-\mu_{\alpha_2}^U\right)^\lambda+\mu_{\alpha_1}^{U^\lambda}\mu_{\alpha_2}^{U^\lambda}}\right],\right.\right.$$

$$\left[\frac{\left(1+\gamma_{\alpha_1}^L\right)^\lambda\left(1+\gamma_{\alpha_2}^L\right)^\lambda-\left(1-\gamma_{\alpha_1}^L\right)^\lambda\left(1-\gamma_{\alpha_2}^L\right)^\lambda}{\left(1+\gamma_{\alpha_1}^L\right)^\lambda\left(1+\gamma_{\alpha_2}^L\right)^\lambda+\left(1-\gamma_{\alpha_1}^L\right)^\lambda\left(1-\gamma_{\alpha_2}^L\right)^\lambda},\frac{\left(1+\gamma_{\alpha_1}^U\right)^\lambda\left(1+\gamma_{\alpha_2}^U\right)^\lambda-\left(1-\gamma_{\alpha_1}^U\right)^\lambda\left(1-\gamma_{\alpha_2}^U\right)^\lambda}{\left(1+\gamma_{\alpha_1}^U\right)^\lambda\left(1+\gamma_{\alpha_2}^U\right)^\lambda+\left(1-\gamma_{\alpha_1}^U\right)^\lambda\left(1-\gamma_{\alpha_2}^U\right)^\lambda}\right]\Bigg)\Bigg/\alpha_1\in\tilde{h}_1,\alpha_2\in\tilde{h}_2\Bigg\}\rightarrow(4)$$

From (3) and (4)

$$\left(h_1\otimes h_2\right)^\lambda=h_1^\lambda\otimes h_2^\lambda$$

To Prove: $(\lambda_1 \oplus \lambda_2)\tilde{h} = \lambda_1\tilde{h} \oplus \lambda_2\tilde{h}$

Let $\tilde{h} = \left\{\left[\mu_\alpha^L, \mu_\alpha^U\right], \left[\gamma_\alpha^L, \gamma_\alpha^U\right] / \alpha \in \tilde{h}\right\}$

$$(\lambda_1+\lambda_2)\tilde{h} = \left\{\left(\left(\left[\frac{(1+\mu_\alpha^L)^{\lambda_1+\lambda_2}-(1-\mu_\alpha^L)^{\lambda_1+\lambda_2}}{(1+\mu_\alpha^L)^{\lambda_1+\lambda_2}+(1-\mu_\alpha^L)^{\lambda_1+\lambda_2}}, \frac{(1+\mu_\alpha^U)^{\lambda_1+\lambda_2}-(1-\mu_\alpha^U)^{\lambda_1+\lambda_2}}{(1+\mu_\alpha^U)^{\lambda_1+\lambda_2}+(1-\mu_\alpha^U)^{\lambda_1+\lambda_2}},\right]\right.\right.\right.$$

$$\left.\left[\frac{2(\gamma_\alpha^L)^{\lambda_1+\lambda_2}}{(2-\gamma_\alpha^L)^{\lambda_1+\lambda_2}+(\gamma_\alpha^L)^{\lambda_1+\lambda_2}}, \frac{2(\gamma_\alpha^U)^{\lambda_1+\lambda_2}}{(2-\gamma_\alpha^U)^{\lambda_1+\lambda_2}+(\gamma_\alpha^U)^{\lambda_1+\lambda_2}}\right] \Big/ \alpha \in \tilde{h}, \lambda_1, \lambda_2 > 0\right\} \rightarrow (5)$$

$$\lambda_1\tilde{h} \oplus \lambda_2\tilde{h} = \left\{\left(\left(\left[\frac{(1+\mu_\alpha^L)^{\lambda_1}-(1-\mu_\alpha^L)^{\lambda_1}}{(1+\mu_\alpha^L)^{\lambda_1}+(1-\mu_\alpha^L)^{\lambda_1}}, \frac{(1+\mu_\alpha^U)^{\lambda_1}-(1-\mu_\alpha^U)^{\lambda_1}}{(1+\mu_\alpha^U)^{\lambda_1}+(1-\mu_\alpha^U)^{\lambda_1}}\right],\right.\right.\right.$$

$$\left.\left.\left.\left[\frac{2\gamma_\alpha^{L^{\lambda_1}}}{(2-\gamma_\alpha^L)^{\lambda_1}+(\gamma_\alpha^L)^{\lambda_1}}, \frac{2\gamma_\alpha^{U^{\lambda_1}}}{(2-\gamma_\alpha^U)^{\lambda_1}+(\gamma_\alpha^U)^{\lambda_1}}\right]\right)\right) \Big/ \alpha \in \tilde{h}\right\} \oplus$$

$$\left\{\left(\left(\left[\frac{(1+\mu_\alpha^L)^{\lambda_2}-(1-\mu_\alpha^L)^{\lambda_2}}{(1+\mu_\alpha^L)^{\lambda_2}+(1-\mu_\alpha^L)^{\lambda_2}}, \frac{(1+\mu_\alpha^U)^{\lambda_2}-(1-\mu_\alpha^U)^{\lambda_2}}{(1+\mu_\alpha^U)^{\lambda_2}+(1-\mu_\alpha^U)^{\lambda_2}}\right],\right.\right.\right.$$

$$\left.\left.\left.\left[\frac{2\gamma_\alpha^{L^{\lambda_2}}}{(2-\gamma_\alpha^L)^{\lambda_2}+(\gamma_\alpha^L)^{\lambda_2}}, \frac{2\gamma_\alpha^{U^{\lambda_2}}}{(2-\gamma_\alpha^U)^{\lambda_2}+(\gamma_\alpha^U)^{\lambda_2}}\right]\right)\right) \Big/ \alpha \in \tilde{h}\right\}$$

$$= \left\{\left(\left(\left[\frac{\frac{(1+\mu_\alpha^L)^{\lambda_1}-(1-\mu_\alpha^L)^{\lambda_1}}{(1+\mu_\alpha^L)^{\lambda_1}+(1-\mu_\alpha^L)^{\lambda_1}}+\frac{(1+\mu_\alpha^L)^{\lambda_2}-(1-\mu_\alpha^L)^{\lambda_2}}{(1+\mu_\alpha^L)^{\lambda_2}+(1-\mu_\alpha^L)^{\lambda_2}}}{1+\frac{(1+\mu_\alpha^L)^{\lambda_1}-(1-\mu_\alpha^L)^{\lambda_1}}{(1+\mu_\alpha^L)^{\lambda_1}+(1-\mu_\alpha^L)^{\lambda_1}}\times\frac{(1+\mu_\alpha^L)^{\lambda_2}-(1-\mu_\alpha^L)^{\lambda_2}}{(1+\mu_\alpha^L)^{\lambda_2}+(1-\mu_\alpha^L)^{\lambda_2}}},\right.\right.\right.\right.$$

$$\left.\frac{\frac{(1+\mu_\alpha^U)^{\lambda_1}-(1-\mu_\alpha^U)^{\lambda_1}}{(1+\mu_\alpha^U)^{\lambda_1}+(1-\mu_\alpha^U)^{\lambda_1}}+\frac{(1+\mu_\alpha^U)^{\lambda_2}-(1-\mu_\alpha^U)^{\lambda_2}}{(1+\mu_\alpha^U)^{\lambda_2}+(1-\mu_\alpha^U)^{\lambda_2}}}{1+\frac{(1+\mu_\alpha^U)^{\lambda_1}-(1-\mu_\alpha^U)^{\lambda_1}}{(1+\mu_\alpha^U)^{\lambda_1}+(1-\mu_\alpha^U)^{\lambda_1}}\times\frac{(1+\mu_\alpha^U)^{\lambda_2}-(1-\mu_\alpha^U)^{\lambda_2}}{(1+\mu_\alpha^U)^{\lambda_2}+(1-\mu_\alpha^U)^{\lambda_2}}}\right],$$

$$\left[\frac{\dfrac{2\gamma_{\alpha}^{L^{\lambda_1}}}{\left(2-\gamma_{\alpha}^{L}\right)^{\lambda_1}+\left(\gamma_{\alpha}^{L}\right)^{\lambda_1}}\times\dfrac{2\gamma_{\alpha}^{L^{\lambda_2}}}{\left(2-\gamma_{\alpha}^{L}\right)^{\lambda_2}+\left(\gamma_{\alpha}^{L}\right)^{\lambda_2}}}{1+\left(1-\dfrac{2\gamma_{\alpha}^{L^{\lambda_1}}}{\left(2-\gamma_{\alpha}^{L}\right)^{\lambda_1}+\left(\gamma_{\alpha}^{L}\right)^{\lambda_1}}\right)\left(1-\dfrac{2\gamma_{\alpha}^{L^{\lambda_2}}}{\left(2-\gamma_{\alpha}^{L}\right)^{\lambda_2}+\left(\gamma_{\alpha}^{L}\right)^{\lambda_2}}\right)},\right.$$

$$\left.\frac{\dfrac{2\gamma_{\alpha}^{U^{\lambda_1}}}{\left(2-\gamma_{\alpha}^{U}\right)^{\lambda_1}+\left(\gamma_{\alpha}^{U}\right)^{\lambda_1}}\times\dfrac{2\gamma_{\alpha}^{U^{\lambda_2}}}{\left(2-\gamma_{\alpha}^{U}\right)^{\lambda_2}+\left(\gamma_{\alpha}^{U}\right)^{\lambda_2}}}{1+\left(1-\dfrac{2\gamma_{\alpha}^{U^{\lambda_1}}}{\left(2-\gamma_{\alpha}^{U}\right)^{\lambda_1}+\left(\gamma_{\alpha}^{L}\right)^{\lambda_1}}\right)\left(1-\dfrac{2\gamma_{\alpha}^{U^{\lambda_2}}}{\left(2-\gamma_{\alpha}^{U}\right)^{\lambda_2}+\left(\gamma_{\alpha}^{U}\right)^{\lambda_2}}\right)}\right]\Bigg)\Bigg/\alpha\in\tilde{h}\Bigg\}$$

$$=\left\{\left(\left(\left[\frac{\left(A^{L^{\lambda_1}}-B^{L^{\lambda_1}}\right)\left(A^{L^{\lambda_2}}+B^{L^{\lambda_2}}\right)+\left(A^{L^{\lambda_1}}+B^{L^{\lambda_1}}\right)\left(A^{L^{\lambda_2}}-B^{L^{\lambda_2}}\right)}{\left(A^{L^{\lambda_1}}+B^{L^{\lambda_1}}\right)\left(A^{L^{\lambda_2}}+B^{L^{\lambda_2}}\right)-\left(A^{L^{\lambda_1}}-B^{L^{\lambda_1}}\right)\left(A^{L^{\lambda_2}}-B^{L^{\lambda_2}}\right)},\right.\right.\right.\right.$$

$$\left.\frac{\left(A^{U^{\lambda_1}}-B^{U^{\lambda_1}}\right)\left(A^{U^{\lambda_2}}+B^{U^{\lambda_2}}\right)+\left(A^{U^{\lambda_1}}+B^{U^{\lambda_1}}\right)\left(A^{U^{\lambda_2}}-B^{U^{\lambda_2}}\right)}{\left(A^{U^{\lambda_1}}+B^{U^{\lambda_1}}\right)\left(A^{U^{\lambda_2}}+B^{U^{\lambda_2}}\right)-\left(A^{U^{\lambda_1}}-B^{L^{\lambda_1}}\right)\left(A^{U^{\lambda_2}}-B^{U^{\lambda_2}}\right)}\right],$$

$$\left[\frac{2\left(C^{L}\right)^{\lambda_1}\square 2\left(C^{L}\right)^{\lambda_2}}{\left(\left(2-C^{L}\right)^{\lambda_1}+\left(C^{L}\right)^{\lambda_1}\right)\left(\left(2-C^{L}\right)^{\lambda_2}+\left(C^{L}\right)^{\lambda_2}\right)+\left(\left(2-C^{L}\right)^{\lambda_1}+\left(C^{L}\right)^{\lambda_1}\right)\left(\left(2-C^{L}\right)^{\lambda_2}+\left(C^{L}\right)^{\lambda_2}\right)},\right.$$

$$\left.\left.\left.\frac{2\left(C^{U}\right)^{\lambda_1}\square 2\left(C^{U}\right)^{\lambda_2}}{\left(\left(2-C^{U}\right)^{\lambda_1}+\left(C^{U}\right)^{\lambda_1}\right)\left(\left(2-C^{U}\right)^{\lambda_2}+\left(C^{U}\right)^{\lambda_2}\right)+\left(\left(2-C^{U}\right)^{\lambda_1}+\left(C^{U}\right)^{\lambda_1}\right)\left(\left(2-C^{U}\right)^{\lambda_2}+\left(C^{U}\right)^{\lambda_2}\right)}\right]\right)\right)\right\}$$

where $A^L = 1+\mu_{\alpha}^{L}$, $A^U = 1+\mu_{\alpha}^{U}$, $B^L = 1-\mu_{\alpha}^{L}$, $B^U = 1-\mu_{\alpha}^{U}$, $C^L = \gamma_{\alpha}^{L}$ and $C^U = \gamma_{\alpha}^{U}$

$$=\left\{\left(\left(\left[\frac{\left(\left(A^{L}\right)^{\lambda_1+\lambda_2}-\left(B^{L}\right)^{\lambda_1+\lambda_2}\right)}{\left(\left(A^{L}\right)^{\lambda_1+\lambda_2}+\left(B^{L}\right)^{\lambda_1+\lambda_2}\right)},\frac{\left(\left(A^{U}\right)^{\lambda_1+\lambda_2}-\left(B^{U}\right)^{\lambda_1+\lambda_2}\right)}{\left(\left(A^{U}\right)^{\lambda_1+\lambda_2}+\left(B^{U}\right)^{\lambda_1+\lambda_2}\right)}\right],\right.\right.\right.$$

$$\left.\left.\left.\left[\frac{2\left(C^{L}\right)^{\lambda_1+\lambda_2}}{\left(\left(2-C^{L}\right)^{\lambda_1+\lambda_2}+\left(C^{L}\right)^{\lambda_1+\lambda_2}\right)},\frac{2\left(C^{U}\right)^{\lambda_1+\lambda_2}}{\left(\left(2-C^{U}\right)^{\lambda_1+\lambda_2}+\left(C^{U}\right)^{\lambda_1+\lambda_2}\right)}\right]\right)\right)\right\}$$

$$=\left\{\left(\left[\frac{\left(1+\mu_\alpha^L\right)^{\lambda_1+\lambda_2}-\left(1-\mu_\alpha^L\right)^{\lambda_1+\lambda_2}}{\left(1+\mu_\alpha^L\right)^{\lambda_1+\lambda_2}+\left(1-\mu_\alpha^L\right)^{\lambda_1+\lambda_2}},\frac{\left(1+\mu_\alpha^U\right)^{\lambda_1+\lambda_2}-\left(1-\mu_\alpha^U\right)^{\lambda_1+\lambda_2}}{\left(1+\mu_\alpha^U\right)^{\lambda_1+\lambda_2}+\left(1-\mu_\alpha^U\right)^{\lambda_1+\lambda_2}}\right],\right.\right.$$

$$\left.\left.\left[\frac{2{\gamma_\alpha^L}^{\lambda_1+\lambda_2}}{\left(2-\gamma_\alpha^L\right)^{\lambda_1+\lambda_2}+\left(\gamma_\alpha^L\right)^{\lambda_1+\lambda_2}},\frac{2{\gamma_\alpha^U}^{\lambda_1+\lambda_2}}{\left(2-\gamma_\alpha^U\right)^{\lambda_1+\lambda_2}+\left(\gamma_\alpha^U\right)^{\lambda_1+\lambda_2}}\right]\right)/\alpha\in\tilde{h},\lambda_1,\lambda_2>0\right\} \quad (6)$$

From equations (5) and (6) $\left(\lambda_1\oplus\lambda_2\right)\tilde{h} = \lambda_1\tilde{h}\oplus\lambda_2\tilde{h}$.

Similarly we can prove that $\tilde{h}^{\lambda_1}\otimes\tilde{h}^{\lambda_2} = \tilde{h}^{(\lambda_1+\lambda_2)}$.

4. INTERVAL VALUED INTUITIONISTIC HESITANT FUZZY GEOMETRIC AGGREGATION OPERATORS BASED ON EINSTEIN OPERATIONS

In this section we develop some geometric aggregation operators based on IVIHFSs

Definition 4.1

Let $\tilde{h}_i=\left\{\left(\left[\mu_{\alpha_i}^L,\mu_{\alpha_i}^U\right],\left[\gamma_{\alpha_i}^L,\gamma_{\alpha_i}^U\right]\right)/\alpha_i\in\tilde{h}_i\right\}$ be a set of IVIHFSs in L , the lattice of non-empty intervals

$L=\left\{[a,b]/(a,b)\in[0,1]^2\right\}$ with partial ordering $\leq_L$. If $\omega = \left(\omega_1,\omega_2,...,\omega_n\right)^T$ is the weight vector of $\tilde{h}_i$ (i=1,2,…,n) such that $\omega_i\in[0,1]$ with $\sum_{i=1}^{n}\omega_i=1$,then an interval valued intuitionistic hesitant fuzzy Einstein weighted geometric (IVIHFWG$^\epsilon$) operators of dimension n is a mapping IVIHFWG$^\epsilon$: L^n → L defined as $\mathbf{IVIHFWG}^\varepsilon\left(\tilde{h}_1,\tilde{h}_2,...,\tilde{h}_n\right)=\overset{n}{\underset{i=1}{\oplus}}\omega_i\tilde{h}_i$.

If $\omega = \left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right)^T$ then IVIHFWG$^\epsilon$ reduces to interval valued intuitionistic hesitant fuzzy Einstein geometric (IVIHFG$^\epsilon$) operator of dimension n. i.e., $\mathbf{IVIHFG}^\varepsilon\left(\tilde{h}_1,\tilde{h}_2,...,\tilde{h}_n\right)=\left(\tilde{h}_1\times\tilde{h}_2\times...\times\tilde{h}_n\right)^{1/n}$.

Based on Einstein operations we state the following theorem.

Theorem 4.1: Let $\tilde{h}_i$ $(i=1,2,...,n)$ be a collection of IVIHFEs. Then their aggregated value by using IVIHFWG$^\epsilon$ operator is also an IVIHFE and

$$\textbf{IVIHFWG}_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,...,\tilde{h}_n\right)=\left\{\left(\left[\frac{2\prod_{i=1}^{n}\left(\mu_{\alpha_i}^{L}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\mu_{\alpha_i}^{L}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\mu_{\alpha_i}^{L}\right)^{\omega_i}},\frac{2\prod_{i=1}^{n}\left(\mu_{\alpha_i}^{U}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\mu_{\alpha_i}^{U}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\mu_{\alpha_i}^{U}\right)^{\omega_i}}\right],\right.\right.$$

$$\left.\left.\left[\frac{\prod_{i=1}^{n}\left(1+\gamma_{\alpha_i}^{L}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\gamma_{\alpha_i}^{L}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\gamma_{\alpha_i}^{L}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\gamma_{\alpha_i}^{L}\right)^{\omega_i}},\frac{\prod_{i=1}^{n}\left(1+\gamma_{\alpha_i}^{U}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\gamma_{\alpha_i}^{U}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\gamma_{\alpha_i}^{U}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\gamma_{\alpha_i}^{U}\right)^{\omega_i}}\right]\right)\Big/\alpha_i\in\tilde{h}_i,(i=1,2,...,n)\right\}$$

Where $\omega=\left(\omega_1,\omega_2,...,\omega_n\right)^T$ is the weight vector of $\tilde{h}_i$ (i=1,2,…,n) such that $\omega_i\in[0,1]$ with $\sum_{i=1}^{n}\omega_i=1$.

If $\mu_{\alpha_i}^{L}=\mu_{\alpha_i}^{U}=\mu_{\alpha_i}$ and $\gamma_{\alpha_i}^{L}=\gamma_{\alpha_i}^{U}=\gamma_{\alpha_i}$ for all $(i=1,2,...,n)$, then the IVIHFWG$^{\epsilon}$ reduces to Intuitionistic Hesitant fuzzy Einstein weighted geometric IHFWG$^{\epsilon}$ operator and the IVIHFWG$^{\epsilon}$ operators satisfies some desirable properties such as Idempotency, Boundedness and Monotonicity.

Definition 4.3

An OWA operator of dimension n is a mapping $f: R^n\rightarrow R$ such that f[$a_1,a_2…,a_n$] = $\sum_{j=1}^{n}\omega_j b_j$ where b_j is the j^{th} largest of the a_i and ω_j is the weight of b_j which satisfies $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1$.

Based on the definition of OWA operator we propose a type of interval valued intuitionistic hesitant fuzzy Einstein ordered weighted geometric (IVIHFOWG$^{\epsilon}$) operator.

Definition 4.4

Let $\tilde{h}_i=\left\{\left(\left[\mu_{\alpha_i}^{L},\mu_{\alpha_i}^{U}\right],\left[\gamma_{\alpha_i}^{L},\gamma_{\alpha_i}^{U}\right]\right)/\alpha_i\in\tilde{h}_i\right\}$ $(i=1,2,...,n)$ be a collection of IVIHFSs in L.

An IVIHFOWG$^{\epsilon}$ operator of dimension n is a mapping from $L^n\rightarrow L$ such that $IVIHFOWG_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,...,\tilde{h}_n\right)=\tilde{h}_{\sigma(1)}^{\omega_1}\times\tilde{h}_{\sigma(2)}^{\omega_2}\times...\times\tilde{h}_{\sigma(n)}^{\omega_n}$ where $\left(\sigma(1),\sigma(2),...,\sigma(n)\right)$ is a permutation of (1,2,…,n) such that IVIHFE $\tilde{h}_{\sigma(i)}^{\omega_1}\le\tilde{h}_{\sigma(i-1)}^{\omega_2}$ for all i = 2,3,…,n. $\omega=\left(\omega_1,\omega_2,...,\omega_n\right)^T$ is the weight vector of $\tilde{h}_i$ (i=1,2,…,n) such that $\omega_i\in[0,1]$ with $\sum_{i=1}^{n}\omega_i=1$.

Based on the Einstein operations we state the following theorem.

Theorem 4.2

Let $\tilde{h}_i=\left\{\left(\left[\mu_{\alpha_i}^{L},\mu_{\alpha_i}^{U}\right],\left[\gamma_{\alpha_i}^{L},\gamma_{\alpha_i}^{U}\right]\right)/\alpha_i\in\tilde{h}_i\right\}$ $(i=1,2,...,n)$ be a collection of IVIHFSs in L. Then their aggregated value by using the IVIHFOWG$^{\epsilon}$ operator is also an IVIHFS an

$$IVIHFOWG_{\omega}^{\varepsilon}\left(\tilde{h}_1,\tilde{h}_2,...,\tilde{h}_n\right)=\left\{\left(\left(\left[\frac{2\prod_{i=1}^{n}\left(\mu_{\sigma(i)}^{L}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\mu_{\sigma(i)}^{L}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\mu_{\sigma(i)}^{L}\right)^{\omega_i}},\frac{2\prod_{i=1}^{n}\left(\mu_{\sigma(i)}^{U}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(2-\mu_{\sigma(i)}^{U}\right)^{\omega_i}+\prod_{i=1}^{n}\left(\mu_{\sigma(i)}^{U}\right)^{\omega_i}}\right],\right.\right.\right.$$

$$\left.\left.\left.\left[\frac{\prod_{i=1}^{n}\left(1+\gamma_{\sigma(i)}^{L}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\gamma_{\sigma(i)}^{L}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\gamma_{\sigma(i)}^{L}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\gamma_{\sigma(i)}^{L}\right)^{\omega_i}},\frac{\prod_{i=1}^{n}\left(1+\gamma_{\sigma(i)}^{U}\right)^{\omega_i}-\prod_{i=1}^{n}\left(1-\gamma_{\sigma(i)}^{U}\right)^{\omega_i}}{\prod_{i=1}^{n}\left(1+\gamma_{\sigma(i)}^{U}\right)^{\omega_i}+\prod_{i=1}^{n}\left(1-\gamma_{\sigma(i)}^{U}\right)^{\omega_i}}\right]\right)\right)\middle/\alpha_i\in\tilde{h}_i,(i=1,2,...,n)\right\}$$

where $(\sigma(1),\sigma(2),...,\sigma(n))$ is a permutation of (1, 2, …, n) such that IVIHFE $\tilde{h}_{\sigma(i)}^{\omega_1}\leq\tilde{h}_{\sigma(i-1)}^{\omega_2}$ for all i = 2,3,…,n. $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ is the weight vector of $\tilde{h}_i$ (i=1,2,…,n) such that $\omega_i\in[0,1]$ with $\sum_{i=1}^{n}\omega_i=1$.

If $\mu_{\sigma(i)}^{L}=\mu_{\sigma(i)}^{U}=\mu_{\sigma(i)}$ and $\gamma_{\sigma(i)}^{L}=\gamma_{\sigma(i)}^{U}=\gamma_{\sigma(i)}$ for all $(i=1,2,...,n)$, then the IVIHFOWG$^{\epsilon}$ reduces to Intuitionistic Hesitant fuzzy Einstein ordered weighted geometric IHFOWG$^{\epsilon}$ operator and the IVIHFWG$^{\epsilon}$ operators satisfies some desirable properties such as Idem potency, Boundedness and Montonicity.

5. CONCLUSIONS

Although many techniques have been introduced to aggregate intuitionistic fuzzy information, very few interval valued Intuitionistic hesitant fuzzy aggregation techniques exist in literature. We have defined several new operators of IVIHFEs such as Einstein sum, Einstein product, Einstein scalar multiplication since Einstein *t*-norm typically gives the same smooth approximations as product *t*-norm. In this paper some new aggregation operators, such as the IVIHFWG$^{\epsilon}$ and IVIHFOWG$^{\epsilon}$ operators are developed based on these Einstein operations to accommodate the Interval valued intuitionistic hesitant fuzzy situations. Various properties of these operators are also investigated.

In future, we will apply the proposed aggregation operators to some real life MADM applications.

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